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Gauge-invariant description of the electromagnetic field in the Higgs phase of the Standard Model

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Abstract

It is shown that the definition of the photon field in the broken-symmetry phase of the electroweak theory proposed recently [7] is equivalent to that proposed previously in Ref. [2].

1 Introduction

In all models of particle physics, electromagnetism appears as part of a larger symmetry group. For example, in standard electroweak theory the photon is a "mixture" of the $SU(2)$ gauge boson and the hypercharge field; in grand unified theories, the $U(1)$ group is a subgroup of a simple unifying group, such as $SU(5)$, $SO(10)$, etc. The ordinary electric ($\vec{E}$) and magnetic ($\vec{B}$) fields are directly observable and thus could be expressed through initial gauge and scalar fields in a gauge-invariant way. The 't Hooft expression [1] for the electromagnetic tensor for $SO(3)$ gauge theory with triplet of scalar fields, in which symmetry is broken down to $U(1)$ (Glashow-Georgi model) is a famous example. This expression is essential for the analysis of the magnetic monopole properties.

This type of analysis has been applied to the case of the electroweak theory [2], where a gauge-invariant expression for the electromagnetic field was constructed. Recently, the problem of gauge-invariant definition of the electromagnetic field in the standard model has attracted a lot of attention in connection with the possibility of primordial magnetic field generation at the electroweak phase transition. Such a definition appeared in Ref. [3] to show that the electromagnetic field could be produced, provided the vacuum expectation value of the Higgs field is not constant during the phase transition. In Ref. [4], another definition is used to argue that the magnetic fields may arise from some semiclassical configurations of the gauge fields, such as Z-strings and W-condensates. However, these definitions [3, 4] have led to some paradoxes. For example,

they impliy that a magnetic field would always be present along the internal axis of the electroweak string. This is resolved using the definition of the electromagnetic tensor [6] that had appeared previously in the study of the electroweak sphaleron [5] and for the Glashow-Georgi $SO(3)$ model [1]. The advantages of that definition are discussed at length in Refs. [6, 7]. One of them is that, in the absence of monopoles, there is no magnetic charge or magnetic current, so there are no contributions to the electromagnetic tensor from electrically neutral currents.

As well known from the Maxwell theory, an electromagnetic tensor can be defined from a vector field that undergoes a gauge transformation by the addition of the gradient of an arbitrary scalar function. Such a vector field was recently constructed from initial scalar and gauge fields in the electroweak theory [7]. In this note, we will show that the results of Ref. [7] are in fact equivalent to the original definition of Ref. [2].

2 Construction of the photon field

With the gauge group $SU(2)\times U(1)$, the construction of the photon field given in Ref. [2] may be summarised by choosing a four-dimensional representation of scalar fields,

$$\boldsymbol{\varphi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix} , \tag{1}$$

and the corresponding $SU(2)$ generators :

$$T^1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \end{pmatrix} , \qquad T^2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ -\mathrm{i} & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \end{pmatrix} ,$$

$$T^3 = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & -\mathrm{i} & 0 \end{pmatrix} , \tag{2}$$

with respective gauge fields A^1, A^2, A^3 and the $U(1)$ generator

$$Y = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \end{pmatrix} \tag{3}$$

with gauge field A^0. The canonical conjugate fields to $\boldsymbol{\varphi}$, A^a_r, A^0_r are respectively $\mathbf{p}$, B^a_r, B^0_r. Let us consider new fields given by the canonical transformation

$$(\boldsymbol{\varphi},\ \mathbf{p},\ A^a_r,\ B^a_r,\ A^0_r,\ B^0_r) \ \longrightarrow\ (\rho,\ p_\rho,\ \theta^a,\ p^a_\theta,\ \widetilde{A}^a_r,\ \widetilde{B}^a_r,\ \widetilde{A}^0_r,\ \widetilde{B}^0_r) \tag{4}$$

where

$$\boldsymbol{\varphi} = \exp(2\mathrm{i}\ \theta^a T^a) \begin{pmatrix} 0 \\ 0 \\ 0 \\ \rho \end{pmatrix} \equiv U(\boldsymbol{\varphi})^\dagger \begin{pmatrix} 0 \\ 0 \\ 0 \\ \rho \end{pmatrix}, \qquad \rho = |\boldsymbol{\varphi}|\ , \tag{5}$$

$$p_\rho = \frac{1}{\rho}\ {}^t\mathbf{p}\ \boldsymbol{\varphi}\ , \tag{6}$$

$$\widetilde{\widehat{A}}_r = U(\boldsymbol{\varphi})\Big(\widehat{A}_r - \frac{1}{\mathrm{i}\ g}\ \partial_r\Big)U(\boldsymbol{\varphi})^\dagger\ , \tag{7}$$

$$\widetilde{\widehat{B}}_r = U(\boldsymbol{\varphi})\ \widehat{B}_r\ U(\boldsymbol{\varphi})^\dagger\ , \tag{8}$$

$$\widetilde{A}^0_r = A^0_r\ , \tag{9}$$

$$\widetilde{B}^0_r = B^0_r\ . \tag{10}$$

Here, $\widehat{A}_r \equiv \sum_{a=1}^3 A^a_r T^a$, $r = 1, 2, 3$, and from (5)

$$U(\boldsymbol{\varphi}) = \frac{1}{\rho} \begin{pmatrix} \varphi_4 & \varphi_3 & -\varphi_2 & -\varphi_1 \\ -\varphi_3 & \varphi_4 & \varphi_1 & -\varphi_2 \\ \varphi_2 & -\varphi_1 & \varphi_4 & -\varphi_3 \\ \varphi_1 & \varphi_2 & \varphi_3 & \varphi_4 \end{pmatrix}\ . \tag{11}$$

The new vector fields are not gauge-invariant :

$$\widetilde{\widehat{A}}_r \quad \longrightarrow \quad \exp(\mathrm{i}\ \lambda_0 T^3)\Big(\widetilde{\widehat{A}}_r - \frac{1}{\mathrm{i}\ g}\ \partial_r\Big)\exp(-\mathrm{i}\ \lambda_0 T^3)\ , \tag{12}$$

$$\widetilde{A}^0_r \quad \longrightarrow \quad \widetilde{A}^0_r + \frac{1}{g\prime}\ \partial_r\lambda_0 \tag{13}$$

(under a gauge transformation, $\Lambda = \exp(\mathrm{i}\ \lambda^a T^a + \mathrm{i}\ \lambda_0 Y)$ with λ^1, λ^2, λ^3, λ_0 arbitrary real numbers). It can be seen from (12) that the field $\widetilde{A}^3$ is only transformed by a gradient stretching :

$$\widetilde{A}^3_r \ \longrightarrow\ \widetilde{A}^3_r + \frac{1}{g}\ \partial_r\lambda_0\ , \qquad r = 1,\ 2,\ 3. \tag{14}$$

Thus we have two fields which are transformed by the addition of a pure gradient. The photon field is then given by the linear combination of $\widetilde{A}^0$ and $\widetilde{A}^3$,

$$E_r = \frac{g\ \widetilde{A}^0_r\ +\ g\prime\ \widetilde{A}^3_r}{\sqrt{g^2 + g\prime^2}}\ , \qquad r = 1,\ 2,\ 3, \tag{15}$$

because it transforms by a pure gradient stretching

$$E_r \ \longrightarrow\ E_r + \frac{1}{e}\ \partial_r\lambda_0\ , \qquad r = 1,\ 2,\ 3, \tag{16}$$

where $e = \frac{g\ g\prime}{\sqrt{g^2 + g\prime^2}}$.

3 The photon field expressed in terms of the initial fields

The generators T^a, $a = 1,\ 2,\ 3$, satisfy the orthonormalisation condition $\mathrm{Tr}(T^a T^b) = \delta^{ab}$, so that $\widetilde{A}^3_r = \mathrm{Tr}(\widetilde{\widetilde{A}}_r T^3)$ and, from (7),

$$\begin{aligned} \widetilde{A}^3_r \quad = \quad & A^1_r\ \frac{2}{\rho^2}\ \big(\varphi_1\ \varphi_3 + \varphi_2\ \varphi_4\big) + A^2_r\ \frac{2}{\rho^2}\ \big(\varphi_2\ \varphi_3 + \varphi_1\ \varphi_4\big) + \\ & + A^3_r\ \frac{1}{\rho^2}\ \big(- \varphi_1^2 - \varphi_2^2 + \varphi_3^2 + \varphi_4^2\big) + \\ & - \frac{1}{g}\ \frac{2}{\rho^2}\ \big(\varphi_1\ \partial_r\varphi_2 - \varphi_2\ \partial_r\varphi_1 + \varphi_3\ \partial_r\varphi_4 - \varphi_4\ \partial_r\varphi_3\big)\ . \end{aligned} \tag{17}$$

Using (17) and (9) in (15), E_r can be written in terms of the initial fields :

$$\begin{aligned} E_r \quad = \quad & \frac{g\ A^0_r}{\sqrt{g^2 + g\prime^2}} + \frac{g\prime}{\sqrt{g^2 + g\prime^2}}\ \frac{1}{\rho^2}\ \Big\{2\big(\varphi_1\ \varphi_3 + \varphi_2\ \varphi_4\big)\ A^1_r + \\ & + 2\big(\varphi_2\ \varphi_3 - \varphi_1\ \varphi_4\big)\ A^2_r + \big(- \varphi_1^2 - \varphi_2^2 + \varphi_3^2 + \varphi_4^2\big)\ A^3_r + \\ & - \frac{2}{g}\ \big(\varphi_1\ \partial_r\varphi_2 - \varphi_2\ \partial_r\varphi_1 + \varphi_3\ \partial_r\varphi_4 - \varphi_4\ \partial_r\varphi_3\big)\Big\}\ . \end{aligned} \tag{18}$$

In Ref. [7], the following expression is given for the photon field :

$$A^{\mathrm{em}}_\mu = \cos\theta_{\mathrm{W}}\ A^0_\mu + \sin\theta_{\mathrm{W}}\Big(- \hat{\phi}^a A^a_\mu + \frac{\mathrm{i}}{g}\ \mathrm{Tr}(\sigma^3\ V^\dagger\ \partial_\mu V)\Big) \tag{19}$$

where $\hat{\phi}^a = \frac{\mathbf{\Phi}^\dagger \sigma^a \mathbf{\Phi}}{\mathbf{\Phi}^\dagger\ \mathbf{\Phi}}$, $a = 1, 2, 3$ (σ^a being the Pauli spin matrices), $V = \frac{1}{\mathbf{\Phi}^\dagger\ \mathbf{\Phi}} \begin{pmatrix} \phi_2^\star & \phi_1 \\ -\phi_1^\star & \phi_2 \end{pmatrix}$ and $\mathbf{\Phi} = (\phi_1, \phi_2)$ is the complex scalar doublet representation. Defining

$$\phi_1 \quad = \quad \varphi_1 - \mathrm{i}\ \varphi_2\ , \tag{20}$$

$$\phi_2 \quad = \quad -\varphi_3 + \mathrm{i}\ \varphi_4\ , \tag{21}$$

we have explicitly

$$\hat{\phi}^1 = \frac{1}{\mathbf{\Phi}^\dagger\ \mathbf{\Phi}}\ \big(\phi_1^\star\ \phi_2 + \phi_1\ \phi_2^\star\big) = -\frac{2}{\rho^2}\ \big(\varphi_1\ \varphi_3 + \varphi_2\ \varphi_4\big)\ , \tag{22}$$

$$\hat{\phi}^2 = \frac{\mathrm{i}}{\mathbf{\Phi}^\dagger\ \mathbf{\Phi}}\ \big(- \phi_1^\star\ \phi_2 + \phi_1\ \phi_2^\star\big) = -\frac{2}{\rho^2}\ \big(\varphi_2\ \varphi_3 - \varphi_1\ \varphi_4\big)\ , \tag{23}$$

$$\hat{\phi}^3 = \frac{1}{\mathbf{\Phi}^\dagger\ \mathbf{\Phi}}\ \big(\phi_1^\star\ \phi_1 - \phi_2^\star\ \phi_2\big) = \frac{1}{\rho^2}\ \big(\varphi_1^2 + \varphi_2^2 - \varphi_3^2 - \varphi_4^2\big)\ , \tag{24}$$

$$\begin{aligned} \mathrm{Tr}\big(\sigma^3\ V^\dagger\ \partial_\mu V\big) &= \frac{1}{\mathbf{\Phi}^\dagger\ \mathbf{\Phi}}\ \big(\phi_2\ \partial_\mu\phi_2^\star + \phi_1\ \partial_\mu\phi_1^\star - \phi_1^\star\ \partial_\mu\phi_1 - \phi_2^\star\ \partial_\mu\phi_2\big) \\ &= \frac{2\ \mathrm{i}}{\rho^2}\ \big(\varphi_1\ \partial_\mu\varphi_2 - \varphi_2\ \partial_\mu\varphi_1 + \varphi_3\ \partial_\mu\varphi_4 - \varphi_4\ \partial_\mu\varphi_3\big)\ . \end{aligned} \tag{25}$$

Substiting these expressions in (19),

$$\begin{aligned} A^{\mathrm{em}}_\mu \quad = \quad & \frac{g\ A^0_r}{\sqrt{g^2 + g\prime^2}} + \frac{g\prime}{\sqrt{g^2 + g\prime^2}}\ \frac{1}{\rho^2}\ \Big\{2\big(\varphi_1\ \varphi_3 + \varphi_2\ \varphi_4\big)\ A^1_\mu + \\ & + 2\big(\varphi_2\ \varphi_3 + \varphi_1\ \varphi_4\big)\ A^2_\mu + \big(- \varphi_1^2 - \varphi_2^2 + \varphi_3^2 + \varphi_4^2\big)\ A^3_\mu + \\ & - \frac{2}{g}\ \big(\varphi_1\ \partial_\mu\varphi_2 - \varphi_2\ \partial_\mu\varphi_1 + \varphi_3\ \partial_\mu\varphi_4 - \varphi_4\ \partial_\mu\varphi_3\big)\Big\}\ . \end{aligned} \tag{26}$$

This corresponds to our result (18).

4 Conclusion

In the Higgs phase of the standard model, it is possible to construct a vector field which, under a gauge transformation, undergoes the addition of a pure gradient. The method is that proposed in Ref. [2], where a four-dimensional real representation of scalar fields has been used. We then computed the photon field in terms of the initial scalar and gauge fields. Such an expression has also appeared in Ref. [7], for a complex scalar doublet representation. We have shown that, using an appropriate parametrisation of the complex fields by real fields, the two formulations are equivalent.

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